ACADEMIC PRESS

J. Math. Anal. Appl. 272 (2002) 448–457

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

Existence of solution for Eguchi–Oki–Matsumura equation describing phase separation and order–disorder transition in binary alloys

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Received 12 March 2001

Submitted by H. Levine

Dedicated to Professor Tetsuo Eguchi on his eightieth birthday

Abstract

In this paper we consider the Eguchi–Oki–Matsumura equation which consists of the fourth- and second-order coupled equations of parabolic type. It is shown that this system admits the unique global solution.

1. Introduction

In phase transition it is well known that the phase separation is described by Cahn-Hilliard equation [5], while the order–disorder transition by Allen–Cahn equation [2]. Mathematical studies for these equations have been fully developed (see [4,14,15]). Recently, there increases the interest for the evolution of coexistent phase in alloy systems that exhibit simultaneous phase separation and one or

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0022-247X/02/$ – see front matter
PII: S0022-247X(02)00154-3

more order–disorder transition. In describing the composition variations at least two length scales are necessary, one on an atomic scale in the ordered domains and the other on the scale of distances between phase and domain boundaries. Obviously it is not reasonable to expect to find a continuum description based on the mean concentration as the sole dominant variables.

In 1984, Eguchi et al. [7] overcame this difficulty and firstly derived the model equation for the simplest case. Later, Cahn and Novick-Cohen [6] derived model equation for simultaneous order–disorder phase separation in binary alloys on a BCC lattice in the neighbourhood of the triple points. In more detail, time-dependent Ginzburg–Landau theory leads that the motion for the relaxation from the nonequilibrium state is described by equation

$$\begin{cases} \frac{\partial u}{\partial t} = L(\theta)\nabla^2 \frac{\delta F}{\delta u}, \\ \frac{\partial v}{\partial t} = -M(\theta)\nabla \frac{\delta F}{\delta v}. \end{cases}$$

Here u is the local concentration of the solute atoms, v is the local degree of order, $L(\theta)$ and $M(\theta)$ are the positive reaction rates depending upon temperature θ, and F is the free energy functional with the bulk free energy $f(u,v)$,

$$F = \int \left(f(u,v) + \frac{h(\theta)}{2}|\nabla u|^2 + \frac{k(\theta)}{2}|\nabla v|^2 \right) dx,$$

where $h(\theta)$ and $k(\theta)$ represent the interfacial energies per unit length along the boundaries of changing the concentration and the degree of order, respectively. Assuming that the order–disorder transformation is of second order and that the phase separation cannot take place in the disordered state, but can in the ordered state, Eguchi et al. [7] (see also [12,13]) adopted the bulk energy $f(u,v)$ in the form

$$f_e(u,v) = A(\theta)\left(u^2 - \frac{X_0(\theta)^2}{2}v^2 + \frac{1}{4}X_1(\theta)^2 v^4 + \frac{1}{2}u^2v^2 \right),$$

where $A(\theta)$ is a positive constant depending on the temperature θ, and $X_0(\theta)$ and $X_1(\theta)$ give the order–disorder transition line with second order and the phase boundary between the mixed and the ordered phase fields, respectively. Novick-Cohen [11] (see also [6]) introduced the following form of $f(u,v)$ as a quasicontinuum limit of the discrete bulk energy:

$$f_c(u,v) = \frac{\theta}{2}\big(g(u+v) + g(u-v)\big) + \alpha u(1-u) - \beta v^2,$$
$$g(s) = s\log s + (1-s)\log(1-s).$$

In their model it can be described for a first-order phase separation.

When we consider the special phase transition that Eguchi et al. argued, phenomenologically it seems to be sufficient to use $f_e(u,v)$ as shown in [12, 13], and mathematically $f_e(u,v)$ is the quartic polynomial of a Taylor expansion of $f_c(u,v)$ and has no singularity so that it can be treated easier than $f_c(u,v)$.

After a suitable normalization, we lead to the initial–boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta(-\Delta u + 2u + uv^2), \quad x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} = \beta\Delta v + \alpha v(a^2 - u^2 - b^2v^2), \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, \quad x \in \Gamma,\ t > 0, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ $(n = 1, 2, 3)$ with smooth boundary Γ. Here $u(x,t)$ is the local concentration of the solute atoms, $v(x,t)$ is the local degree of order, α, β, a and b are positive constants, and $\partial/\partial n$ is the exterior normal derivative to Γ.

The aim of this paper is to establish existence and uniqueness theorems to problem (1).

Theorem 1.1. *For any* $(u_0, v_0) \in H^2(\Omega)$ *satisfying the compatibility conditions*

$$\frac{\partial u_0}{\partial n}\bigg|_{\Gamma} = \frac{\partial v_0}{\partial n}\bigg|_{\Gamma} = 0,$$

problem (1) *has a unique solution* (u, v) *defined on* $Q_{T'} \equiv \Omega \times (0, T')$ *for some* $T' > 0$ *such that*

$$u \in H^{4,1}(Q_{T'}) \cap C\big(0, T'; H^2(\Omega)\big),$$
$$v \in L^2\big(0, T'; H^3(\Omega)\big) \cap H^1\big(0, T'; L^2(\Omega)\big) \cap C\big(0, T'; H^2(\Omega)\big).$$

Here $H^{4,1}(Q_T) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega))$.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *problem* (1) *admits a unique global solution* (u, v) *on* Q_T *for any* $T > 0$.

Similar results for Cahn–Hilliard equation was established by Elliot and Zheng [8]. One-dimensional stationary problem to (1) was discussed in [9]. Finite-dimensional exponential attractor for the system similar to (1) was constructed in [3].

The organization of this paper is as follows: the local existence is discussed in Section 2 and global one in Section 3.

2. Existence of the local solution

In this section, we discuss the local existence to initial–boundary value problem (1). Since $(d/dt)\int_\Omega u(x,t)\,dx = 0$, it is convenient to change a variable from

u to $u+\bar{u}$ in (1), where $\bar{u}\equiv(1/|\Omega|)\int_\Omega u(x,t)\,dx=(1/|\Omega|)\int_\Omega u_0(x)\,dx$. Then we obtain

$$\begin{cases} \frac{\partial u}{\partial t}=\Delta(-\Delta u+2u+(u+\bar{u})v^2), & x\in\Omega,\ t>0,\\ \frac{\partial v}{\partial t}=\beta\Delta v+\alpha v(a^2-(u+\bar{u})^2-b^2v^2), & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega,\\ \frac{\partial u}{\partial n}=\frac{\partial\Delta u}{\partial n}=0,\quad \frac{\partial v}{\partial n}=0, & x\in\Gamma,\ t>0. \end{cases} \tag{2}$$

We begin with preparing auxiliary lemmas.

Lemma 2.1 [14, Chapter 3, Lemma 4.2]. *For $u\in H^4(\Omega)$ satisfying $(\partial u/\partial n)|_\Gamma=(\partial\Delta u/\partial n)|_\Gamma=0$ and $\int_\Omega u(x)\,dx=0$, the norms $\|u\|_{H^4(\Omega)}$ and $\|\Delta^2u\|=\|\Delta^2u\|_{L^2(\Omega)}$ are equivalent. Similarly, $\|v\|_{H^2(\Omega)}$ and $(\|\Delta v\|^2+\|v\|^2)^{1/2}$ are equivalent for $v\in H^2(\Omega)$ with $(\partial v/\partial n)|_\Gamma=0$.*

The linearized problem of (2) is discussed in [10, Chapter 4] (see also [4, Chapter 6]).

Lemma 2.2. *Suppose that $(u_0,v_0)\in H^2(\Omega)$ satisfies the compatibility conditions and Δf, g, $\Delta g\in L_2(0,T;L^2(\Omega))$ with any $T>0$. Then problem*

$$\begin{cases} \frac{\partial u}{\partial t}+\Delta^2u=\Delta f(x,t), & (x,t)\in Q_T,\\ \frac{\partial v}{\partial t}-\beta\Delta v=g(x,t), & (x,t)\in Q_T,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega,\\ \frac{\partial u}{\partial n}=\frac{\partial\Delta u}{\partial n}=0,\quad \frac{\partial v}{\partial n}=0, & (x,t)\in\Gamma_T\equiv\Gamma\times(0,T), \end{cases} \tag{3}$$

has a unique solution

$$\begin{aligned} &u\in H^{4,1}(Q_T)\cap C\big(0,T;H^2(\Omega)\big),\\ &v\in L^2\big(0,T;H^3(\Omega)\big)\cap H^1\big(0,T;L^2(\Omega)\big)\cap C\big(0,T;H^2(\Omega)\big), \end{aligned} \tag{4}$$

which satisfies estimate

$$\begin{aligned} \|(u,v)\|_{X_T}^2\equiv{}&\sup_{0\leqslant t\leqslant T}\|(u,v)(t)\|_{H^2(\Omega)}^2\\ &+\int_0^T\big(\|(u_t,v_t)(s)\|^2+\|u(s)\|_{H^4(\Omega)}^2+\|v(s)\|_{H^3(\Omega)}^2\big)\,ds\\ \leqslant{}& C_1e^{C_2T}\Bigg(\|(u_0,v_0)\|_{H^2(\Omega)}^2\\ &+\int_0^T\big(\|\Delta f(s)\|^2+\|g(s)\|^2+\|\Delta g(s)\|^2\big)\,ds\Bigg) \end{aligned} \tag{5}$$

for some positive constants C_1 and C_2 independent of T.

Let us proceed to the proof of Theorem 1.1, which will be carried out by the contraction mapping principle. We introduce the space

$$X_T = \left\{ (u,v) \;\middle|\; \begin{array}{l} (u,v) \text{ satisfies (4)}, \ \int_\Omega u\,dx = \int_\Omega u_0\,dx = 0, \\ \frac{\partial u}{\partial n}\big|_{\Gamma_T} = \frac{\partial \Delta u}{\partial n}\big|_{\Gamma_T} = 0, \ \frac{\partial v}{\partial n}\big|_{\Gamma_T} = 0, \\ \|(u,v)\|_{X_T}^2 \leqslant 2C_1 \|(u_0,v_0)\|_{H^2(\Omega)}^2 \end{array} \right\}.$$

For a given $(\varphi,\psi) \in X_T$, we consider linear problem

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u = \Delta(2\varphi + (\varphi+\bar{u})\psi^2) \equiv \Delta f(\varphi,\psi), & (x,t) \in Q_T, \\ \frac{\partial v}{\partial t} - \beta\Delta v = \alpha\psi(a^2 - (\varphi+\bar{u})^2 - b^2\psi^2) \equiv g(\varphi,\psi), & (x,t) \in Q_T, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, & (x,t) \in \Gamma_T. \end{cases} \tag{6}$$

From Lemma 2.2, there exists a unique solution (u,v) to problem (6) satisfying (4) and (5). Some elementary calculations together with inequalities

$$\|u\|_{L_\infty(\Omega)} \leqslant c\|u\|_{H^2(\Omega)}, \tag{7}$$

$$\|\nabla u\|_{L_4(\Omega)} \leqslant c\|u\|_{H^2(\Omega)} \tag{8}$$

yield estimate

$$\int_0^T \left(\|\Delta f(s)\|^2 + \|g(s)\|^2 + \|\Delta g(s)\|^2\right) ds \leqslant C_3 \|(u_0,v_0)\|_{H^2(\Omega)}^2 T, \tag{9}$$

where the constant C_3 depends on $\|(u_0,v_0)\|_{H^2(\Omega)}$ nondecreasingly. Hence if we take the number $T_1 \in (0,T)$ so small that

$$e^{C_2 T_1}(1 + C_3 T_1) \leqslant 2, \tag{10}$$

then we find that the mapping $(\varphi,\psi) \to (u,v)$ from X_{T_1} into itself.

Next we consider the equation for $(U,V) \equiv (u_1 - u_2, v_1 - v_2)$,

$$\begin{cases} \frac{\partial U}{\partial t} + \Delta^2 U = \Delta f(\varphi_1,\psi_1) - \Delta f(\varphi_2,\psi_2) \equiv \Delta F, & (x,t) \in Q_{T_1}, \\ \frac{\partial V}{\partial t} - \beta\Delta V = \Delta g(\varphi_1,\psi_1) - \Delta g(\varphi_2,\psi_2) \equiv G, & (x,t) \in Q_{T_1}, \\ U(x,0) = 0, \quad V(x,0) = 0, & x \in \Omega, \\ \frac{\partial U}{\partial n} = \frac{\partial \Delta U}{\partial n} = 0, \quad \frac{\partial V}{\partial n} = 0, & (x,t) \in \Gamma_{T_1}, \end{cases} \tag{11}$$

where $(\varphi_i,\psi_i) \in X_{T_1}$ $(i = 1,2)$. Applying Lemma 2.2, we obtain

$$\|(U,V)\|_{X_{T_1}}^2 \leqslant C_1 e^{C_2 T_1} \int_0^{T_1} \left(\|\Delta F(s)\|^2 + \|G(s)\|^2 + \|\Delta G(s)\|^2\right) ds. \tag{12}$$

Similarly to (9), using (7) and (8) leads to

$$\int_0^{T_1} \left(\|\Delta F(s)\|^2 + \|G(s)\|^2 + \|\Delta G(s)\|^2\right) ds$$
$$\leqslant C_4 \|(\varphi_1 - \varphi_2, \psi_1 - \psi_2)\|_{X_{T_1}}^2 T_1. \tag{13}$$

Here the constant $C_4 > 0$ has the same property as C_3. Therefore, by taking the number $T' \in (0, T_1)$ small enough that

$$C_4 T' < 1, \tag{14}$$

the contraction mapping principle yields the existence of a unique solution $(u, v) \in X_{T'}$ to problem (2). This completes the proof of Theorem 1.1. □

3. A priori estimates and global existence

Theorem 1.2 will be proved by a usual combination of the local existence and the a priori estimates. According to the standard arguments, in this section we can assume that (u, v) is sufficiently smooth. We first note that problem (2) has the Lyapunov functional

$$J(u, v) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{\beta}{2\alpha}|\nabla v|^2 - \frac{a^2}{2}v^2 + \frac{b^2}{4}v^4 + u^2 + \frac{1}{2}(u + \bar{u})^2 v^2\right) dx, \tag{15}$$

which satisfies

$$\frac{d}{dt} J(u, v) + \int_\Omega \left(|\nabla K(u, v)|^2 + \frac{1}{\alpha}|v_t|^2\right) dx = 0 \tag{16}$$

with $K(u, v) \equiv -\Delta u + 2u + (u + \bar{u})v^2$. Therefore we have

Lemma 3.1. *If* (u, v) *satisfies* (2)*, then*

$$\frac{1}{2}\|\nabla u\|^2 + \|u\|^2 + \frac{\beta}{2\alpha}\|\nabla v\|^2 + \frac{b^2}{8}\|v\|_{L^4(\Omega)} + \frac{1}{2}\|(u + \bar{u})v\|^2$$
$$+ \int_0^t ds \int_\Omega \left(|\nabla K(u, v)(s)|^2 + \frac{1}{\alpha}|v_t(s)|^2\right) dx$$
$$\leqslant J(u_0, v_0) + \frac{a^4}{2b^2}|\Omega| \equiv C. \tag{17}$$

Moreover, we can obtain the boundedness of $\|v(t)\|_{L^\infty(\Omega)}$ [1] (see also [4, Chapter 6]).

Lemma 3.2. *Estimate*

$$\sup_{t>0} \|v(t)\|_{L^\infty(\Omega)} \leqslant C \max\Big\{\|v_0\|_{L^\infty(\Omega)}, \sup_{t>0} \|v(t)\|\Big\} \tag{18}$$

is valid for solution (u, v) *to problem* (2). *Here the constant* C *is independent of* T.

We shall use the energy method to obtain further necessary estimates. We prove the case $n = 3$. The cases $n = 1$ and $n = 2$ are easier. We multiply $(2)_1$ by u and integrate over Ω. It follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|\Delta u(t)\|^2 + 2\|\nabla u(t)\|^2 \\
&\quad = \int_\Omega (u + \bar{u})v^2 \Delta u \, dx \leqslant \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|(u + \bar{u})v^2(t)\|^2.
\end{aligned}$$

Using (17) and (18), we get

$$\|u(t)\|^2 + \int_0^t \|\Delta u(s)\|^2 \, ds \leqslant C(T) \quad (0 \leqslant t \leqslant T). \tag{19}$$

Here and in what follows, we denote by $C(T)$ the constants depending on T nondecreasingly, which may change from line to line.

Multiplying $(2)_2$ by Δv and integrating with respect to x, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2 + \beta\|\Delta v(t)\|^2 + \alpha\|(\nabla v)(u + \bar{u})(t)\|^2 + 3\alpha b^2\|v\nabla v(t)\|^2 \\
&\quad = \alpha a^2\|\nabla v(t)\|^2 - 2\alpha \int_\Omega v(u + \bar{u})\nabla u \cdot \nabla v \, dx \\
&\quad \leqslant \alpha a^2\|\nabla v(t)\|^2 + \frac{\alpha}{2}\|(\nabla v)(u + \bar{u})(t)\|^2 + 2\alpha\|v\nabla u(t)\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\nabla v(t)\|^2 + \int_0^t \big(\|\Delta v(s)\|^2 + \|(\nabla v)(u + \bar{u})(s)\|^2\big) \, ds \leqslant C(T) \\
&\quad (0 \leqslant t \leqslant T).
\end{aligned} \tag{20}$$

We multiply $(2)_1$ by Δu and integrate over Ω. It follows

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2+\|\nabla\Delta u(t)\|^2+2\|\Delta u(t)\|^2\\
&\quad=\int_\Omega \nabla\big((u+\bar{u})v^2\big)\cdot\nabla\Delta u\,dx\leqslant\frac{1}{2}\|\nabla\Delta u(t)\|^2+\frac{1}{2}\big\|\nabla\big((u+\bar{u})v^2\big)(t)\big\|^2.
\end{aligned}
$$

Since

$$
\big\|\nabla\big((u+\bar{u})v^2\big)(t)\big\|^2\leqslant C\big(\|(\nabla u)v^2(t)\|^2+\|(u+\bar{u})v\nabla v(t)\|^2\big),
$$

Lemmas 3.1, 3.2 and (20) yield that

$$
\|\nabla u(t)\|^2+\int_0^t\|\nabla\Delta u(s)\|^2\,ds\leqslant C(T)\quad(0\leqslant t\leqslant T). \tag{21}
$$

Now we apply Δ to Eq. $(2)_2$, multiply it by Δv and integrate over Ω. Since it is easy to check from (2) that $\partial\Delta v/\partial n=0$ on Γ_T, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta v(t)\|^2+\beta\|\nabla\Delta v(t)\|^2\\
&\quad=-\alpha\int_\Omega\nabla\big(v\big(a^2-(u+\bar{u})^2-b^2v^2\big)\big)\cdot\nabla\Delta v\,dx\\
&\quad\leqslant\frac{\beta}{2}\|\nabla\Delta v(t)\|^2+\frac{\alpha^2}{2\beta}\big\|\nabla\big(v\big(a^2-(u+\bar{u})^2-b^2v^2\big)\big)(t)\big\|^2.
\end{aligned}
$$

Gagliardo–Nirenberg inequality

$$
\|u\|_{L^\infty(\Omega)}\leqslant C\|\nabla\Delta u\|^{1/2}\|u\|^{1/2}+C'\|u\|
$$

yields

$$
\begin{aligned}
&\big\|\nabla\big(v\big(a^2-(u+\bar{u})^2-b^2v^2\big)\big)(t)\big\|^2\\
&\quad\leqslant C\big(\|\nabla v(t)\|^2+\|(\nabla v)(u+\bar{u})^2(t)\|^2\\
&\qquad+\|v(u+\bar{u})\nabla u(t)\|^2+\|v^2(\nabla v)(t)\|^2\big)\leqslant C(T)\big(\|\nabla\Delta u(t)\|^2+1\big).
\end{aligned}
$$

Hence estimate

$$
\|\Delta v(t)\|^2+\int_0^t\|\nabla\Delta v(s)\|^2\,ds\leqslant C(T)\quad(0\leqslant t\leqslant T) \tag{22}
$$

results.

Multiplying $(2)_1$ by $\Delta^2 u$ and integrating with respect to x imply

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta u(t)\|^2+\|\Delta^2u(t)\|^2+2\|\nabla\Delta u(t)\|^2\\
&\quad\leqslant\frac{1}{2}\|\Delta^2u(t)\|^2+\frac{1}{2}\big\|\Delta\big((u+\bar{u})v^2\big)(t)\big\|^2.
\end{aligned}
$$

For the last term in the above inequality we have

$$\begin{aligned}\bigl\|\Delta\bigl((u+\bar{u})v^2\bigr)(t)\bigr\|^2 \leqslant C\bigl(&\|(\Delta u)v^2(t)\|^2+\|(u+\bar{u})(\nabla v)^2(t)\|^2\\&+\|(u+\bar{u})v\Delta v(t)\|^2+\|(\nabla u)v(\nabla v)(t)\|^2\bigr).\end{aligned}$$

Terms in the right-hand side, for example, are estimated as

$$\begin{aligned}\|u(\nabla v)^2(t)\|^2 &\leqslant \|u(t)\|^2_{L^\infty(\Omega)}\|\nabla v(t)\|^4_{L^4(\Omega)}\\&\leqslant C(T)\bigl(\|\Delta u(t)\|^{3/2}+1\bigr)\bigl(\|\nabla\Delta v(t)\|^{1/3}+1\bigr),\end{aligned}$$

$$\begin{aligned}\|(\nabla u)v(\nabla v)(t)\|^2 &\leqslant \|\nabla u(t)\|^2_{L^\infty(\Omega)}\|v(t)\|^2_{L^\infty(\Omega)}\|\nabla v(t)\|^2\\&\leqslant C(T)\bigl(\|\nabla\Delta u(t)\|^{5/3}+1\bigr).\end{aligned}$$

Therefore, we conclude that

$$\|\Delta u(t)\|^2+\int_0^t\|\Delta^2u(s)\|^2\,ds \leqslant C(T)\quad(0\leqslant t\leqslant T). \tag{23}$$

Finally, it follows from $(2)_1$ that

$$\int_0^t\|u_t(s)\|^2\,ds \leqslant C(T)\quad(0\leqslant t\leqslant T). \tag{24}$$

Lemmas 3.1, 3.2 and estimates (19)–(24) together with Theorem 1.1 prove Theorem 1.2. The proof is completed. □

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